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Lax pairs of time-dependent Gross–Pitaevskii equation

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Abstract
We calculate the Lax pairs of homogeneous and inhomogeneous one-dimensional time-dependent Gross–Pitaevskii equations with time-dependent scattering length. The inhomogeneity corresponds to linear and quadratic potentials. Our approach introduces a systematic method of searching for the Lax pair corresponding to a given differential equation. We derive known Lax pairs for the Gross–Pitaevskii equation with homogeneous and quadratic potentials and time-dependent scattering length. We also derive new Lax pairs corresponding to a Gross–Pitaevskii equation with a linear potential. Using the resulting Lax pairs, the Darboux transformation can be performed and exact solutions of the Gross–Pitaevskii equation can be obtained for experimentally relevant cases such as solitonic solutions.

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1. Introduction

The experimental realization of atomic Bose–Einstein condensates in confining potentials [1] and the use of Feschbach resonance to tune the interatomic interactions [2] led to the realization of *dark* and *bright* solitons in these condensates [3–7]. Studying the behaviour of these solitons requires solving an inhomogeneous nonlinear Schrödinger equation known as the Gross–Pitaevskii equation [8]. Although numerical solutions are available [9], it will be more interesting and insightful to have exact analytical solutions of this equation. This will be of particular importance to the investigation of the dynamics of vortices and solitons [10, 11]. Exact solitonic solutions are known for the homogeneous one-dimensional Gross–Pitaevskii equation [12]. These solutions account for the dark and bright solitons realized in the experiments. The difficulty is in solving the Gross–Pitaevskii equation with the presence of the inhomogeneity arising from the potential that traps the condensate. Nonetheless, it has been shown by Liang *et al* [13] that the method of *Darboux transformation* [14] can be used to obtain exact solutions of the Gross–Pitaevskii equation with quadratic expulsive potential and scattering length that is growing exponentially with time.

The Darboux transformation, or analogously *Bäcklund or dressing transformation*, applies only to systems of linear differential equations and cannot be applied directly to nonlinear differential equations. To be able to apply the Darboux transformation to a certain nonlinear differential equation, one finds a linear system of equations that is equivalent to a nonlinear differential equation. The relation between the linear system and the nonlinear differential equation is established through a *consistency condition* satisfied by the linear system. The Darboux transformation is then applied to the linear system resulting in transforming the equivalent nonlinear equation as well. The linear system is usually represented in terms of a pair of matrices called the *Lax pair* which must satisfy a consistency condition that is equivalent to the differential equation at hand. The difficulty is usually in finding this Lax pair. It is known for some nonlinear differential equations such as the Kortweg–de Vries (KdV) equation, the sine-Gordon equation and the nonlinear Schrödinger equation [14]. In addition to the Lax pair, one also needs to know an exact solution of the nonlinear differential equation. This exact solution is then used as a *seed* for the Darboux transformation to generate other exact solutions.

In the present work, we present a method of obtaining these two essential ingredients for performing the Darboux transformation, namely the Lax pair and the seed solution. In addition to some known Lax pairs for the Gross–Pitaevskii equation, our method successfully produces ones. In addition to the interest in constructing new nontrivial solitonic solutions from simpler ones, there is also interest in obtaining generalized Lax pairs of modified Gross–Pitaevskii equations concerning integrable and chaotic behaviour [15–17].

The rest of the paper is organized as follows. In section 2, we present the Gross–Pitaevskii equation. In section 3, we review the Darboux transformation method. In section 4, we calculate the Lax pair. In section 5, we derive seed solutions to the Gross–Pitaevskii equation. Section 6 contains a discussion and conclusions.

2. The Gross–Pitaevskii equation

The mean-field equation of motion governing the evolution of the wavefunction of the Bose–Einstein condensate is the so-called time-dependent Gross–Pitaevskii equation [8]

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t)+\frac{1}{2}m\left(\omega_x^2x^2+\omega_y^2y^2+\omega_z^2z^2\right)\psi(\mathbf{r},t)+g|\psi(\mathbf{r},t)|^2\psi(\mathbf{r},t), \quad (1)$$

which is basically a Schrödinger equation but with a nonlinear term. Here, g is the effective two-particle interaction which is proportional to the s-wave scattering length a according to $g=4\pi a\hbar^2/m$, where m is the mass of an atom, and ω_x, ω_y and ω_z are the trap frequencies in the x, y and z directions, respectively. For axially symmetric elongated traps, where the confining along, say, the y and z directions is much stronger than along the x direction, namely $\omega_y=\omega_z=\omega_\perp\gg\omega_x$, the condensate is quasi one dimensional. The Gross–Pitaevskii equation can then be integrated over the transverse directions to reduce to a one-dimensional nonlinear Schrödinger equation [12]

$$i\hbar\frac{\partial\psi(x,t)}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}+\frac{1}{2}m\omega_x^2x^2\psi(x,t)+\frac{g}{2\pi a_\perp^2}|\psi(x,t)|^2\psi(x,t), \quad (2)$$

where $a_\perp=\sqrt{\hbar/m\omega_\perp}$ and $\omega_\perp$ are the characteristic length and frequency of the harmonic trap in the transverse direction, respectively. Scaling length to $a_\perp$, time to $2/\omega_\perp$ and $\psi(x,t)$

to $1/\sqrt{2a_\perp}$, the last equation takes the dimensionless form

$$\mathrm{i}\frac{\partial\psi(x,t)}{\partial t} = -\frac{\partial^2\psi(x,t)}{\partial x^2} + \frac{1}{4}\lambda^2x^2\psi(x,t) + 2a|\psi(x,t)|^2\psi(x,t), \tag{3}$$

where $\lambda = 2\omega_x/\omega_\perp$.

In the homogeneous case, $\lambda = 0$, exact bright and dark solitonic solutions can be obtained for attractive interatomic interactions, $a < 0$, and repulsive interactions, $a > 0$, respectively [12]. For an expulsive harmonic potential, it was shown by Liang *et al* [13] that an exact solitonic solution can also be obtained provided that the scattering length is given by $a(t) = a_0 \exp(\lambda t)$, where a_0 is the scattering length at $t = 0$. In this case, the Gross–Pitaevskii equation takes the form

$$\mathrm{i}\frac{\partial\psi(x,t)}{\partial t} + \frac{\partial^2\psi(x,t)}{\partial x^2} + \frac{1}{4}\lambda^2x^2\psi(x,t) + 2a_0\,\mathrm{e}^{\lambda t}|\psi(x,t)|^2\psi(x,t) = 0. \tag{4}$$

In the present work, we attempt to find the Lax pair of a Gross–Pitaevskii of the general form

$$\mathrm{i}\frac{\partial\psi(x,t)}{\partial t} + \frac{\partial^2\psi(x,t)}{\partial x^2} + \frac{1}{4}\lambda(x)^2\psi(x,t) + 2a(t)|\psi(x,t)|^2\psi(x,t) = 0, \tag{5}$$

where $a(t) = a_0\,\mathrm{e}^{\gamma(t)t}$ and $\lambda(x)$ and $\gamma(t)$ are assumed to be independent general functions of x and t, respectively. For the special case of $\lambda(x) = \lambda x$ and $\gamma(t) = \lambda$, the expulsive potential case, equation (4) are retrieved.

3. The Darboux transformation and the Lax pair

Darboux transformation is a method used to obtain exact solutions of nonlinear partial differential equations [14]. In this method, a linear system of equations for a field $\Phi(x,t)$ is written in the form

$$\Phi_x = U\Phi, \tag{6}$$

$$\Phi_t = V\Phi, \tag{7}$$

where the matrices, U and V, are called the Lax pair. The subscripts x and t denote, here and throughout, position and time derivatives, respectively. The consistency condition $\Phi_{xt} = \Phi_{tx}$ requires that U and V obey

$$U_t - V_x + [U, V] = 0. \tag{8}$$

The Lax pair is expressed in terms of the wavefunction $\psi(x,t)$ such that the consistency condition, equation (8), is equivalent to the Gross–Pitaevskii equation. Once the Lax pair is obtained, the Darboux transformation can be applied to the linear system (6) and (7). In the Darboux transformation method, one of the solutions of the linear system, denoted by Φ_1, is chosen as a *seed* to perform the following functional transform on the field Φ:

$$\Phi[1] = \Phi\Lambda - \sigma\Phi, \tag{9}$$

where $\Phi[1]$ is the transformed field, Λ is a constant diagonal matrix and $\sigma = \Phi_1\Lambda\Phi_1^{-1}$. For the system (6) and (7) to be *covariant* with respect to the Darboux transformation, the Lax pair must also be transformed such that

$$\Phi[1]_x = U[1]\Phi[1], \tag{10}$$

$$\Phi[1]_t = V[1]\Phi[1], \tag{11}$$

is satisfied. As a result, the consistency condition is also covariant under the Darboux transformation and takes the form of equation (8) but with the Lax pair, U and V, being replaced by the transformed Lax pair $U[1]$ and $V[1]$. This means that the new wavefunction $\psi[1]$ is also a solution of the Gross–Pitaevskii equation. Thus, using the Darboux transformation, we obtain one exact solution from another. In some cases, this leads to classes of solutions.

The difficulty in this method is in finding the Lax pair that corresponds to a given differential equation. Usually, one starts with a Lax pair and then discovers what differential equation it represents. For the inhomogeneous case, with an expulsive potential the Lax pair corresponding to equation (4) was found to be [13]

$$U = \begin{pmatrix} \zeta & \sqrt{a_0}Q \\ -\sqrt{a_0}Q^* & -\zeta \end{pmatrix}, \tag{12}$$

$$V = \begin{pmatrix} 2\mathrm{i}\zeta^2 + \lambda x\zeta + \mathrm{i}a_0|Q|^2 & \sqrt{a_0}\,[(\lambda x + 2\mathrm{i}\zeta)Q + \mathrm{i}Q_x] \\ \sqrt{a_0}[-(\lambda x + 2\mathrm{i}\zeta)Q^* + \mathrm{i}Q_x^*] & -2\mathrm{i}\zeta^2 - \lambda x\zeta - \mathrm{i}a_0|Q|^2 \end{pmatrix}, \tag{13}$$

Here, $Q(x,t) = \exp(\lambda t/2 - \mathrm{i}\lambda x^2/4)\psi(x,t)$ and $\zeta(t) = \xi_0 \exp(\lambda t)$, where ξ_0 is an arbitrary constant. The consistency condition generates equation (4). The Lax pair of the homogeneous Gross–Pitaevskii equation can be obtained simply by setting $\lambda = 0$ in equations (12) and (13).

Liang *et al* have applied the Darboux transformation on the Lax pair (12) and (13) to derive a solitonic solution of equation (4). Obtaining an exact solution of such a nonlinear partial differential equation is interesting for its own sake. However, the fact that the potential is expulsive, rather than impulsive as in the experiments, and that the time dependence of the scattering length is restricted to an exponential function with a rate that equals the frequency of the trap itself, makes this solution very special and not very useful from a realistic point of view. A question then arises: can we generalize this approach to more realistic cases? Such as having an impulsive harmonic potential and general time dependence of the scattering length independent of the trap frequency? The latter would be of high importance for studying the latest developments in the field including oscillating the optical trap such that the scattering length changes with time sinusoidally [20].

The main aim of this paper is to answer these questions. This is performed in the next section where we expand the Lax pair in powers of the wavefunction $\psi(x,t)$. The unknown coefficients of the expansion are functions of x and t. Then we require that the consistency condition to be equivalent to equation (5). This results in a set of equations for the unknown coefficients. Solving these equations leads to the desired Lax pair that corresponds to the differential equation at hand. However, it turns out that this scheme does not always lead to finding the Lax pair. In some cases, the number of the resulting equations turns out to be larger than the number of the unknown coefficients making the system of equations to be overdetermined. This results in some restrictions on the potentials and time dependences. The fact that, in the work of Liang *et al*, the frequency of the expulsive potential and the rate of exponential time-dependent scattering length are equal is one example on such a restriction.

4. Calculating the Lax pair

In this section, we present our method of obtaining the Lax pair of equation (5). The method is summarized as follows. We calculate the matrix resulting from substituting the Lax pair for the expulsive potential case, (12) and (13), in the consistency condition (8). We modify the resulting matrix such that, when equating it to zero, our general Gross–Pitaevskii equation, equation (5), is obtained. Then we express the unknown Lax pair in terms of second-order polynomials in the wavefunction $\psi(x,t)$ with unknown function coefficients. Calculating the consistency condition using this Lax pair and requiring the result to be equal to the modified matrix, we obtain equations for the unknown coefficients. By solving these equations, we obtain the Lax pair equivalent to equation (5). This approach reproduces the previously found Lax pairs of the expulsive potential and homogeneous potential cases [13, 14], in addition to a new Lax pair for the linear potential case.

Calculating the consistency condition for the expulsive potential case by substituting the Lax pair (12) and (13) into equation (8), we obtain

$$\begin{pmatrix} 0 & -\mathrm{i}\sqrt{a_0}\exp((\lambda t/2+\mathrm{i}\lambda x^2/4))A[\psi,\psi^*] \\ -\mathrm{i}\sqrt{a_0}\exp((\lambda t/2-\mathrm{i}\lambda x^2/4))A^*[\psi,\psi^*] & 0 \end{pmatrix} = 0, \tag{14}$$

where $A[\psi,\psi^*]$ is the left-hand side of equation (4). This leads to $A[\psi,\psi^*]=0$ and $A^*[\psi,\psi^*]=0$ which are the Gross–Pitaevskii equation, equation (4), and its complex conjugate, respectively. The prefactors of $A[\psi,\psi^*]$ and $A^*[\psi,\psi^*]$ result from the functional transformation $Q(x,t)=\exp(\lambda t/2-\mathrm{i}\lambda x^2/4)\psi(x,t)$. In our general case, equation (5) is accounted for if we modify this matrix into

$$\begin{pmatrix} 0 & -\mathrm{i}\sqrt{a_0}\exp(\gamma(t)t/2+\mathrm{i}f(x))B[\psi,\psi^*] \\ -\mathrm{i}\sqrt{a_0}\exp(\gamma(t)t/2-\mathrm{i}f(x))B^*[\psi,\psi^*] & 0 \end{pmatrix} = 0, \tag{15}$$

where $B[\psi,\psi^*]$ is the left-hand side of equation (5). Similar to the expulsive potential case, it turns out that it is more convenient to express the wavefunction $\psi(x,t)$ in terms of the function $Q(x,t)$ as follows:

$$\psi(x,t)=\exp(-\mathrm{i}f(x)-\gamma(t)t/2)Q(x,t). \tag{16}$$

This equation is a generalization of the analogous transformation in the expulsive potential case. The real function $f(x)$ appears only in the prefactors of $B[\psi,\psi^*]$ and $B^*[\psi,\psi^*]$, i.e., the resulting Gross–Pitaevskii equation is independent of $f(x)$. However, it turns out that the Lax pair depends on $f_x(x)$. Thus, we can choose a certain form of $f(x)$ such that the Lax pair is simplified without changing the differential equation that it corresponds to.

We expand the Lax pair in powers of Q, Q_x and their complex conjugates as follows:

$$U=\begin{pmatrix} f_1+f_2Q & f_3+f_4Q \\ f_5+f_6Q^* & f_7+f_8Q^* \end{pmatrix}, \tag{17}$$

$$V=\begin{pmatrix} g_1+g_2Q+g_3Q_x+g_4QQ^* & g_5+g_6Q+g_7Q_x+g_8QQ^* \\ g_9+g_{10}Q^*+g_{11}Q_x^*+g_{12}QQ^* & g_{13}+g_{14}Q^*+g_{15}Q_x^*+g_{16}QQ^* \end{pmatrix}, \tag{18}$$

where $f_{1\text{–}8}(x,t)$ and $g_{1\text{–}16}(x,t)$ are the unknown functions. The matrices U and V are expanded up to the linear and quadratic order, respectively, since their product should account for the cubic term in equation (5). We have terminated the expansion of U at the linear order and excluded many terms from both expansions. This is because when we employed the full expansions of both U and V up to the third order, many of the coefficients vanished since they resulted in terms that are not present in equation (5). For example, having a Q_x-term in U would lead, when calculating the consistency condition, to the term Q_{xt} which is not present in the differential equation. Even in the present form, many of the coefficients of the Lax pair (17) and (18) turn out to be zero. Consequently, the expansions in (17) and (18) are more than sufficient to give rise to the cases of quadratic, linear and homogeneous potentials.

We substitute the Lax pair (17) and (18) in the consistency condition (8) and use equation (16) to express the matrix of equation (15) in terms of $Q(x,t)$ and then we require the left-hand-side of the consistency condition (8) to be equal to the matrix of equation (15). Equating the coefficients of Q, Q_x, Q_{xx}, Q_t, $|Q|^2Q$, and their complex conjugates on both sides of the resulting equation, we obtain 24 equations for the 24 unknown coefficients $f_{1\text{–}8}$ and $g_{1\text{–}16}$. This results in many of the coefficients to be equal to zero or constant, namely: $f_2 = f_3 = f_5 = f_8 = g_2 = g_3 = g_5 = g_8 = g_9 = g_{12} = g_{14} = g_{15} = 0$, $f_4 = -f_6 = \sqrt{a_0}$, $g_7 = g_{11} = \sqrt{a_0}\mathrm{i}$, $g_4 = -g_{16} = a_0\mathrm{i}$. Using these constant values, the equations for the rest of the coefficients simplify to

$$g_{10} = -g_6, \tag{19}$$

$$f_{1t} - g_{1x} = 0, \tag{20}$$

$$f_{7t} - g_{13x} = 0, \tag{21}$$

$$g_{10x} + (f_7 - f_1)g_{10} + \sqrt{a_0}(g_{13} - g_1) - \sqrt{a_0}\left[-\mathrm{i}\lambda^2/4 - \left(\gamma - 2\mathrm{i}f_x^2 + \gamma_t t + 2f_{xx}\right)/2\right] = 0, \tag{22}$$

$$g_{10x} - (f_7 - f_1)g_{10} - \sqrt{a_0}(g_{13} - g_1) + \sqrt{a_0}\left[-\mathrm{i}\lambda^2/4 + \left(\gamma + 2\mathrm{i}f_x^2 + \gamma_t t + 2f_{xx}\right)/2\right] = 0, \tag{23}$$

$$g_{10} + \mathrm{i}\sqrt{a_0}(f_1 - f_7) + 2\sqrt{a_0}f_x = 0. \tag{24}$$

In the following, we solve this system of equations. Adding equations (22) and (23), we find

$$g_{10} = -\sqrt{a_0}(\gamma x + 2f_x + \gamma_t t x)/2 + c_1, \tag{25}$$

where $c_1(t)$ is a constant of integration resulting from integrating over x. Equations (20) and (21) can be both satisfied by assuming $f_7 = \alpha_1 f_1 + \alpha_2$ and $g_{13} = \alpha_1 g_1 + \alpha_3$, where α_1, α_2 and α_3 are the arbitrary constants. For $\alpha_1 = 1$, equations (20) and (21) become equivalent and decouple from equations (22)–(24). Therefore, the solutions of equations (20)–(24) are expected to differ significantly when $\alpha_1 = 1$ in comparison with the $\alpha_1 \neq 1$ solutions. For this reason, we treat below the cases of $\alpha_1 \neq 1$ and $\alpha = 1$ separately.

4.1. Case I: $\alpha_1 \neq 1$

Substituting for g_{10} from equation (25) into equation (24), we get an equation for f_1 whose solution is given by

$$f_1 = \frac{\mathrm{i}}{2(\alpha_1 - 1)}\left(2\mathrm{i}\alpha_2 + \gamma x - \frac{2c_1}{\sqrt{a_0}} - 2f_x + \gamma_t t x\right). \tag{26}$$

Subtracting equations (22) and (23), and using the above expression for f_1, we find

$$g_1 = \frac{-\mathrm{i}}{4(\alpha_1 - 1)a_0}\left[-4c_1^2 + 4\sqrt{a_0}c_1(\gamma + \gamma_t t)x + a_0(\lambda^2 - 4\mathrm{i}\alpha_3 - (\gamma + \gamma_t t)^2 x^2)\right]. \tag{27}$$

Substituting for f_1 and g_1 into equation (20), we find

$$\frac{2}{\sqrt{a_0}}[(\gamma + \gamma_t t)c_1 - c_{1t}] + \lambda\lambda_x + [2\gamma_t - (\gamma + \gamma_t t)^2 + \gamma_{tt} t]x = 0. \tag{28}$$

For this equation to be satisfied, the function $\lambda(x)$ must be of the form

$$\lambda(x) = \sqrt{\lambda_0 + \lambda_1 x + \lambda_2^2 x^2}, \tag{29}$$

where λ_0, λ_1 and λ_2 are the constants. Substituting this expression for $\lambda(x)$ into equation (28), it takes the form

$$\frac{2}{\sqrt{a_0}}\left[\frac{\sqrt{a_0}}{2}\lambda_1 + (\gamma + \gamma_t t)c_1 - c_{1t}\right] + \left[\lambda_2^2 + 2\gamma_t - (\gamma + \gamma_t t)^2 + \gamma_{tt} t\right]x = 0. \tag{30}$$

Here again, the cases of $\lambda_2 \neq 0$ and $\lambda_2 = 0$ should be treated separately.

4.1.1. Case I(a) ($\lambda_2 \neq 0$). Equating separately the first and the second lines of equation (30) to zero, and solving for $\gamma(t)$ and then for $c_1(t)$, we get

$$\gamma(t) = \lambda_2 - \frac{1}{t}\ln(c_2\,\mathrm{e}^{2\lambda_2 t} + c_3), \tag{31}$$

and

$$c_1(t) = \frac{4c_4\lambda_2\,\mathrm{e}^{\lambda_2 t} - \sqrt{a_0}\lambda_1(c_3 - c_2\,\mathrm{e}^{2\lambda_2 t})}{4\lambda_2(c_3 + c_2\,\mathrm{e}^{2\lambda_2 t})}, \tag{32}$$

where c_2, c_3 and c_4 are the constants of integration (independent of x and t). Using the last two equations to substitute for $c_1(t)$ and $\gamma(t)$ into equations (25)–(27), we obtain explicit expressions for $f_1(x,t)$, $g_1(x,t)$ and $g_{10}(x,t)$:

$$f_1(x,t) = \frac{\mathrm{i}\eta_2}{4\lambda_2(\alpha_1 - 1)\eta_1}, \tag{33}$$

$$g_1(x,t) = \frac{\mathrm{i}\left[\left(c_2^2\zeta^4 + c_3^2\right)\eta_4 - 2c_2c_3\zeta^2\eta_5\right]}{16(\alpha_1 - 1)\lambda_2^2\eta_1^2}, \tag{34}$$

$$g_{10}(x,t) = -\frac{\sqrt{a_0}\eta_6}{4\lambda_2\eta_1}, \tag{35}$$

where $\eta_1 = c_3 + c_2\zeta^2$, $\eta_2 = -4\lambda_2 f_x\eta_1 + \left(\lambda_1 + 2\lambda_2^2 x\right)\eta_3$, $\eta_3 = c_3 - c_2\zeta^2$, $\eta_4 = \lambda_1^2 - 4\lambda_0\lambda_2^2$, $\eta_5 = \lambda_1^2 + \lambda_2^2\left(4\lambda_0 + 8\lambda_1 x + 8\lambda_2^2 x^2\right)$, $\eta_6 = 4\lambda_2 f_x\eta_1 + \left(\lambda_1 + 2\lambda_2^2 x\right)\eta_2$ and $\zeta = \exp(\lambda_2 t)$. It should be noted here that since α_2 and α_3 do not appear in the Gross–Pitaevskii equation, they have been readily set to zero. This completes the determination of the unknown coefficient functions $f_{1\text{–}7}$ and $g_{1\text{–}16}$. Substituting for these coefficients into equations (17) and (18), we obtain the Lax pair

$$U = \begin{pmatrix} f_1 & \sqrt{a_0}Q \\ -\sqrt{a_0}Q^* & \alpha_1 f_1 \end{pmatrix}, \tag{36}$$

$$V = \begin{pmatrix} g_1 + \mathrm{i}a_0|Q|^2 & -g_{10}Q + \mathrm{i}\sqrt{a_0}Q_x \\ g_{10}Q^* + \mathrm{i}\sqrt{a_0}Q_x^* & \alpha_1 g_1 - \mathrm{i}a_0|Q|^2 \end{pmatrix}. \tag{37}$$

Calculating the consistency condition (8) using this Lax pair, we obtain the Gross–Pitaevskii equation

$$\mathrm{i}\frac{\partial\psi(x,t)}{\partial t} + \frac{\partial^2\psi(x,t)}{\partial x^2} + \frac{1}{4}\left(\lambda_0 + \lambda_1 x + \lambda_2^2 x^2\right)\psi(x,t) + \frac{2a_0}{c_2\,\mathrm{e}^{\lambda_2 t} + c_3\,\mathrm{e}^{-\lambda_2 t}}|\psi(x,t)|^2\psi(x,t) = 0. \tag{38}$$

The lax pair (17) and (18) corresponding to this Gross–Pitaevskii equation is one of the main results of this paper. It is a generalization to the previous cases of homogeneous potential [14] and quadratic potential [13].

4.1.2. Case I(b) ($\lambda_2 = 0$). In this case, the solution of equation (30) gives

$$\gamma(t) = -\frac{1}{t}\ln(c_2 t + c_3), \tag{39}$$

$$c_1(t) = \frac{8c_4 + \sqrt{a_0}\lambda_1\eta_2 t}{8\eta_1}. \tag{40}$$

The resulting Lax pair is again given by equation (36) with

$$f_1(x,t) = -\frac{\mathrm{i}(4c_2 x + 2c_3\lambda_1 t + c_2\lambda_1 t^2 + 8\eta_1 f_x)}{8(\alpha_1 - 1)\eta_1}, \tag{41}$$

$$g_1(x,t) = \frac{\mathrm{i}\left(4c_3^2\eta_2 + 4c_2c_3\eta_3 t + c_2^2\eta_4\right)}{64(\alpha_1 - 1)\eta_1^2}, \tag{42}$$

$$g_{10}(x,t) = \frac{\sqrt{a_0}}{8\eta_1}(4c_2 x + 2c_3\lambda_1 t + c_2\lambda_1 t^2 - 8\eta_1 f_x). \tag{43}$$

Here, $\eta_1 = c_3 + c_2 t$, $\eta_2 = \lambda_1(\lambda_1 t^2 - 4x) - 4\lambda_0$, $\eta_3 = \eta_2 - 4\lambda_0$ and $\eta_4 = \lambda_1^2 t^4 - 8\lambda_1 x t^2 - 16\lambda_0 t^2 + 16x^2$. The Gross–Pitaevskii equation corresponding to this case is

$$\mathrm{i}\frac{\partial\psi(x,t)}{\partial t} + \frac{\partial^2\psi(x,t)}{\partial x^2} + \frac{1}{4}(\lambda_0 + \lambda_1 x)\psi(x,t) + \frac{2a_0}{c_2 t + c_3}|\psi(x,t)|^2\psi(x,t) = 0. \tag{44}$$

4.2. Case II: $\alpha_1 = 1$

For $\alpha_1 = 1$, equations (20)–(24) reduce to

$$f_{1t} - g_{1x} = 0, \tag{45}$$

$$g_{10x} + \alpha_2 g_{10} + \sqrt{a_0}\left(\alpha_3 + \mathrm{i}\lambda^2/4 + \gamma/2 - \mathrm{i}f_x^2 + \gamma_t t/2 + f_{xx}\right) = 0, \tag{46}$$

$$g_{10x} - \alpha_2 g_{10} - \sqrt{a_0}\left(\alpha_3 + \mathrm{i}\lambda^2/4 - \gamma/2 - \mathrm{i}f_x^2 - \gamma_t t/2 - f_{xx}\right) = 0, \tag{47}$$

$$g_{10} - \mathrm{i}\sqrt{a_0}\alpha_2 + 2\sqrt{a_0}f_x = 0. \tag{48}$$

Subtracting equations (46) and (47) and substituting for g_{10x} using equation (48), we get a second-order differential equation for $f(x)$, with a solution

$$f(x) = (\gamma + \gamma_t t)x^2/4 + c_2 x + c_1, \tag{49}$$

where c_1 and c_2 are the constants. Adding equations (46) and (47) and using the last equation to substitute for $f(x)$, we get

$$g_{10}(x,t) = 4\alpha_3 + \mathrm{i}\lambda(x)^2 - \mathrm{i}(2c_2 + \gamma x + \gamma_t t x)^2. \tag{50}$$

Substituting for $g_{10}(x,t)$ from the last equation and for $f(x)$ from equation (49) into equation (48), we obtain

$$\begin{aligned} &-4\mathrm{i}\alpha_2^2 - 4\alpha_3 + 8\alpha_2 c_2 - \mathrm{i}\left(\lambda_0 - 4c_2^2\right) + [-\mathrm{i}\lambda_1 + 4(\alpha_2 + \mathrm{i}c_2)(\gamma + \gamma_t t)]x \\ &\quad + \left[-\mathrm{i}\left(\lambda_2^2 - \gamma^2 - 2\gamma\gamma_t t - \gamma_t^2 t^2\right)\right]x^2 = 0. \end{aligned} \tag{51}$$

Equating the first line of this equation to zero, we obtain $\alpha_2 = \alpha_3 = 0$ and $c_2 = \sqrt{\lambda_0}/2$. Equating the coefficient of x to zero, we find

$$\gamma(t) = \frac{\lambda_1}{4c_2} + \frac{c_1}{t}, \tag{52}$$

and equating the coefficient of x^2 to zero, we get

$$\gamma(t) = \pm\lambda_2 + \frac{c_1}{t}. \tag{53}$$

Substituting this solution into equation (49), we note that the term $\gamma + \gamma_t t$ vanishes and thus $f(x)$ becomes a function of x only as it should be. The last two equations indicate that $c_2 = \pm\lambda_1/4\lambda_2 = \sqrt{\lambda_0}/2$. Equation (45) is decoupled from equations (46)–(48), and thus can be satisfied by setting an arbitrary expression for $f_1(x,t)$ and then solving for $g_1(x,t)$. Since the choice of $f_1(x,t)$ is arbitrary and will not affect the resulting Gross–Pitaevskii equation, we make the simple choice $f_1(x,t) = g_1(x,t) = 0$. The resulting Lax Pair is given by

$$U = \begin{pmatrix} 0 & \sqrt{a_0}Q \\ -\sqrt{a_0}Q^* & 0 \end{pmatrix}, \tag{54}$$

$$V = \begin{pmatrix} \mathrm{i}a_0|Q|^2 & \sqrt{a_0}[\mathrm{i}Q_x + (\sqrt{\lambda_0} \pm \lambda_2 x)Q] \\ \sqrt{a_0}[\mathrm{i}Q_x^* - (\sqrt{\lambda_0} \pm \lambda_2 x)Q^*] & -\mathrm{i}a_0|Q|^2 \end{pmatrix}, \tag{55}$$

where $Q(x,t) = \exp[(2c_1 + 2\mathrm{i}\sqrt{\lambda_0}x \pm 2\lambda_2 t \pm \mathrm{i}\lambda_2 x^2)/4]\psi(x,t)$. The corresponding Gross–Pitaevskii equation is calculated to be

$$\mathrm{i}\frac{\partial\psi(x,t)}{\partial t} + \frac{\partial^2\psi(x,t)}{\partial x^2} + \frac{1}{4}(\sqrt{\lambda_0} \pm \lambda_2 x)^2\psi(x,t) + 2a_0\,\mathrm{e}^{c_3\pm\lambda_2 t}|\psi(x,t)|^2\psi(x,t) = 0. \tag{56}$$

Substituting $c_2 = 0$ and $f_x(x) = \lambda_2 x/2$ into equations (36) and (37) of Case I ($\alpha_1 \neq 1$), we obtain equations (54) and (56) of the present case with a positive sign of λ_2. For $c_3 = 0$ and $f_x(x) = -\lambda_2 x/2$, we obtain equations (54)–(56) with the negative sign of λ_2. Thus, equations (54)–(56) are just special cases of equations (36), (37) and (38).

4.3. Lax pairs of some special cases

In the following, we use our previous results to calculate the Lax pair for some interesting special cases.

4.3.1. Homogeneous potential ($\lambda(x) = 0$). Setting $\lambda_0 = \lambda_1 = f_x = 0$ in the Lax pair of Case I(b), we get the Lax pair

$$U = \begin{pmatrix} \frac{-\mathrm{i}c_2 x}{2(\alpha_1-1)\eta_1} & \sqrt{a_0}Q \\ -\sqrt{a_0}Q^* & \frac{-\mathrm{i}\alpha_1 c_2 x}{2(\alpha_1-1)\eta_1} \end{pmatrix}, \tag{57}$$

$$V = \begin{pmatrix} \frac{\mathrm{i}c_2^2x^2}{4(\alpha_1-1)\eta_1^2} + \mathrm{i}a_0|Q|^2 & \sqrt{a_0}\left[\mathrm{i}Q_x - \frac{c_2 x}{2\eta_1}Q\right] \\ \sqrt{a_0}\left[\mathrm{i}Q_x^* + \frac{c_2 x}{2\eta_1}Q^*\right] & \frac{\mathrm{i}\alpha_1 c_2^2 x^2}{4(\alpha_1-1)\eta_1^2} - \mathrm{i}a_0|Q|^2 \end{pmatrix}, \tag{58}$$

where $Q(x,t) = \psi(x,t)/\sqrt{c_3 + c_2 t}$, $\eta_1 = c_3 + c_2 t$, c_1, c_2 and α_1 are the arbitrary constants. The Gross–Pitaevskii equation that corresponds to this Lax pair is

$$\mathrm{i}\frac{\partial\psi(x,t)}{\partial t} + \frac{\partial^2\psi(x,t)}{\partial x^2} + \frac{2a_0}{c_3 + c_2 t}|\psi(x,t)|^2\psi(x,t) = 0. \tag{59}$$

Setting $c_2 = 0$ in equations (57)–(59), we reproduce the Lax pair of a Gross–Pitaevskii equation with a zero potential and time-independent scattering length which agrees with the result of [14]. It should be noted that the constant diagonal element of U in [14] would have been accounted for if we have not set α_1 and α_2 to zero.

4.3.2. Linear potential $(\lambda(x)^2 = \lambda_1 x)$. Setting, in Case I(b), $\lambda_0 = f_x = 0$, we get the Lax pair

$$U = \begin{pmatrix} -\frac{i(4c_2x+\lambda_1\eta_2 t)}{8(\alpha_1-1)\eta_1} & \sqrt{a_0}Q \\ -\sqrt{a_0}Q^* & -\frac{i\alpha_1(4c_2x+\lambda_1\eta_2 t)}{8(\alpha_1-1)\eta_1} \end{pmatrix}, \tag{60}$$

$$V = \begin{pmatrix} \frac{i(\lambda_1 t^2-4x)(\lambda_1\eta_2^2-4c_2^2x)}{64(\alpha_1-1)\eta_1^2} + ia_0|Q|^2 & \sqrt{a_0}\left[iQ_x - \frac{\lambda_1\eta_2 t+4c_2x}{8\eta_1}Q\right] \\ \sqrt{a_0}\left[iQ_x^* + \frac{\lambda_1\eta_2 t+4c_2x}{8\eta_1}Q^*\right] & \frac{i\alpha_1(\lambda_1 t^2-4x)(\lambda_1\eta_2^2-4c_2^2x)}{64(\alpha_1-1)\eta_1^2} - ia_0|Q|^2 \end{pmatrix}, \tag{61}$$

where $Q(x,t) = \psi(x,t)/\sqrt{c_3+c_2t}$, $\eta_1 = c_3 + c_2t$ and $\eta_2 = c_3 + \eta_1$. The Gross–Pitaevskii equation that corresponds to this Lax pair is

$$i\frac{\partial\psi(x,t)}{\partial t} + \frac{\partial^2\psi(x,t)}{\partial x^2} + \frac{1}{4}\lambda_1 x\psi(x,t) + \frac{2a_0}{c_3+c_2t}|\psi(x,t)|^2\psi(x,t) = 0. \tag{62}$$

To the best of our knowledge, this is the first time the Lax pair of a Gross–Pitaevskii equation with a linear potential is obtained.

4.3.3. Quadratic expulsive potential $\left(\lambda(x)^2 = \lambda_2^2 x^2\right)$. Setting, in Case I(a), $\lambda_0 = \lambda_1 = 0$, we get the Lax pair

$$U = \begin{pmatrix} \frac{i\lambda_2\eta_2 x}{2(\alpha_1-1)\eta_1} & \sqrt{a_0}Q \\ -\sqrt{a_0}Q^* & \frac{i\alpha_1\lambda_2\eta_2 x}{2(\alpha_1-1)\eta_1} \end{pmatrix}, \tag{63}$$

$$V = \begin{pmatrix} -\frac{ic_2c_3\lambda_2^2\zeta^2x^2}{(\alpha_1-1)\eta_1^2} + ia_0|Q|^2 & \sqrt{a_0}\left[iQ_x + \frac{2\lambda_2\eta_2 x}{2\eta_1}Q\right] \\ \sqrt{a_0}\left[iQ_x^* - \frac{2\lambda_2\eta_2 x}{\eta_1}Q^*\right] & -\frac{i\alpha_1c_2c_3\lambda_2^2\zeta^2x^2}{(\alpha_1-1)\eta_1^2} - ia_0|Q|^2 \end{pmatrix}, \tag{64}$$

where $Q(x,t) = \psi(x,t)/\sqrt{c_3/\zeta + c_2\zeta}$, $\eta_1 = c_3 + c_2\zeta^2$, $\eta_2 = c_3 - c_2\zeta^2$ and $\zeta = \exp(\lambda_2 t)$. The Gross–Pitaevskii equation that corresponds to this Lax pair is

$$i\frac{\partial\psi(x,t)}{\partial t} + \frac{\partial^2\psi(x,t)}{\partial x^2} + \frac{1}{4}\lambda_2^2x^2\psi(x,t) + \frac{2a_0}{c_2\,e^{\lambda_2 t} + c_3\,e^{-\lambda_2 t}}|\psi(x,t)|^2\psi(x,t) = 0. \tag{65}$$

For $c_2 = 0$, $c_3 = 1$ and $f_x = \lambda_2 x/2$, the expulsive potential case, equations (4), (12) and (13), is essentially retrieved.

5. Seed solution

In this section, we derive a solution of the general Gross–Pitaevskii equation, equation (5), to be used as a seed to the Darboux transformation, as described in section 3. This can be achieved by writing $\psi(x,t)$ in the form

$$\psi(x,t) = A\exp[h_1(x,t) + ih_2(x,t)], \tag{66}$$

where A is a real constant and $h_1(x,t)$ and $h_2(x,t)$ are the real functions. Substituting this expression for $\psi(x,t)$ into equation (5), we obtain from the real and imaginary parts

$$8A^2\,e^{2h_1}a_0\gamma(t) + \lambda(x)^2 - 4\left(h_{2t} - h_{1x}^2 + h_{2x}^2 - h_{1xx}\right) = 0, \tag{67}$$

$$h_{1t} + 2h_{1x}h_{2x} + h_{2xx} = 0. \tag{68}$$

We note that with the assumption $h_1(x,t) = h_1(t)$, the last two equations simplify to

$$8A^2\,e^{2h_1}a_0\gamma(t) + \lambda(x)^2 - 4\big(h_{2t} + h_{2x}^2\big) = 0, \tag{69}$$

$$h_{1t} + h_{2xx} = 0. \tag{70}$$

Solving the last equation for $h_2(x,t)$, we obtain

$$h_2(x,t) = c_1(t) + c_2(t)x - \tfrac{1}{2}x^2 h_{1t}. \tag{71}$$

Substituting this expression for h_2 into equation (69), we get

$$8A^2\,e^{2h_1}a_0\gamma(t) + \lambda(x)^2 - 4\big[(c_2 - xh_{1t})^2 + c_{1t} + xc_{2t} - \tfrac{1}{2}x^2 h_{1tt}\big] = 0. \tag{72}$$

This equation can be solved by expanding $\lambda(x)^2$, in powers of x, up to the second order, as in equation (29). Substituting this expansion into equation (72) and equating to zero the coefficients of x^0, x^1 and x^2, we obtain

$$\lambda_0 + 8A^2\,e^{2h_1}a_0\gamma(t) - 4c_2^2 - 4c_{1t} = 0, \tag{73}$$

$$\lambda_1 + 8c_2h_{1t} - 4c_{2t} = 0, \tag{74}$$

$$\lambda_2^2 - 4h_{1t}^2 + 2h_{1tt} = 0. \tag{75}$$

The solution of the last equation is

$$h_1(t) = c_4 - \tfrac{1}{2}\ln\{\cosh[\lambda_2(2c_3 + t)]\} = 0. \tag{76}$$

Substituting this solution into equation (74), we get

$$c_2(t) = c_5\,\mathrm{sech}\,\eta_1 + \frac{\lambda_1}{4\lambda_2}\tanh\eta_1, \tag{77}$$

and equation (73) leads to

$$\begin{aligned} c_1(t) = c_6 + \frac{1}{16\lambda_2^3}\Big\{&\lambda_2(c_7 - t)\big(\lambda_1^2 - 4\lambda_0\lambda_2^2\big) + 8c_5\lambda_1\lambda_2(\mathrm{sech}\,\eta_1 - \mathrm{sech}\,\eta_2)\\ &+ \big(\lambda_1^2 - 16c_5^2\lambda_2^2\big)(\tanh\eta_1 - \tanh\eta_2)\\ &+ 2A^2\,e^{2c_4}g_0\int_{c_7}^{t}\mathrm{d}t'\gamma(t')\,\mathrm{sech}[\lambda_2(2c_3 + t')]\Big\}, \end{aligned} \tag{78}$$

where $\eta_1 = \lambda_2(2c_3 + t)$, $\eta_2 = \lambda_2(2c_3 + c_7)$ and $c_{3\text{–}7}$ are constants of integration.

Having determined the unknown functions $h_1(t)$ and $h_2(x,t)$, the seed solution takes the form

$$\psi(x,t) = A\sqrt{\mathrm{sech}[\lambda_2(2c_3 + t)]}\,\exp\{c_4 + \mathrm{i}[c_1(t) + c_2(t)x + \lambda_2\tanh[\lambda_2(2c_3 + t)]x^2/4]\}, \tag{79}$$

where $c_1(t)$ and $c_2(t)$ are given by equations (78) and (77). Direct substitution of this solution into equation (5) shows that it is indeed a solution for any $a(t)$.

We have seen in the previous section that the Lax pair exists for specific pairs of $\lambda(x)$ and $a(t)$. Using these specific expressions of $a(t)$ in the solution (79), explicit forms of the seed solution can be obtained. For the homogeneous potential case, $\lambda(x) = 0$ and $a(t) = a_0/(c_8 + c_9t)$, the seed solution (79) takes the form

$$\psi(x,t) = A\exp\big[c_4 + 2\mathrm{i}c_6 + \mathrm{i}c_5^2(c_7 - t) + \mathrm{i}c_5x\big]\left[\frac{a(c_7)}{a(t)}\right]^{\frac{2\mathrm{i}A^2a_0\mathrm{e}^{2c_4}}{c_9}}, \tag{80}$$

where c_4–c_7 are arbitrary constants. For the linear potential case, $\lambda(x)^2 = \lambda_1 x$ and $a(t) = a_0/(c_8 + c_9t)$, the seed solution is

$$\psi(x,t) = A\exp\left(\mathrm{i}\lambda_1 tx/4 - \mathrm{i}\lambda_1^2 t^3/48\right)\left[\frac{a(0)}{a(t)}\right]^{\frac{2\mathrm{i}A^2 a_0}{c_9}}, \tag{81}$$

where we have set $c_4 = c_5 = c_6 = c_7 = 0$ for simplicity. Finally, for the quadratic potential case, $\lambda(x)^2 = \lambda_2^2 x^2$ and $a(t) = a_0/\left[c_8\exp(\lambda_2 t) + c_9\exp(-\lambda_2 t)\right]$. The seed solution for this case is given by

$$\psi(x,t) = A\sqrt{\mathrm{sech}(\lambda_2 t)}\exp[\mathrm{i}\lambda_2 x^2\tanh(\lambda_2 t)/4]\left[\frac{a(0)\,\mathrm{sech}(\lambda_2 t)}{a(t)}\right]^{\frac{2\mathrm{i}A^2 a_0}{(c_8-c_9)\lambda_2}}, \tag{82}$$

where here also we have set $c_4 = c_5 = c_6 = c_7 = 0$, for simplicity. Having not set these constants to zero, the seed solutions (81) and (82) turn out to be lengthy and more complicated but may lead to more interesting solitonic solutions when used in the Darboux transformation.

6. Conclusion

The two essential ingredients necessary to solve a differential equation using the Darboux transformation method, namely the Lax pair and the seed solution, have been obtained here for the time-dependent Gross–Pitaevskii equation. Our approach successfully generates the Lax pairs corresponding to the cases of constant, linear and quadratic potentials. This approach can be useful in searching for the Lax pair of other differential equations. A more sophisticated method for finding the Lax pair was developed by Wahlquist and Estabrook [21]. In addition, rather general Lax pairs for general potentials in the Gross–Pitaevskii equation have been constructed [22, 23]. Furthermore, the so-called *inverse scattering method* is also an important tool in constructing nontrivial soliton solutions [22]. Our approach is, on the other hand, more focused on a specific differential equation, namely the Gross–Pitaevskii equation, and on a specific type of solutions, namely the solitonic solutions, and is not meant to be rigorous. The calculation is simple and generates interesting solutions, some of which are new.

For the homogeneous case, the Gross–Pitaevskii equation, equation (59), describes a homogeneous Bose gas with time-dependent interatomic scattering length. The time-independent analogue of this equation (obtained by setting $c_2 = 0$ in equation (59)) has bright and dark soliton solutions for negative and positive interatomic scattering lengths, respectively [12]. These solutions have been studied extensively in the literature where the Darboux transformation was used to generate classes of solitonic solutions [14].

For the linear potential case, the Gross–Pitaevskii equation, equation (62), describes a Bose gas trapped in a linear trap. Although, experimental traps produce quadratic potentials, it has been shown that the surface of a Bose–Einstein condensate and its excitations are described by a Gross–Pitaevskii equation with a linear potential [18, 24]. Using the seed solution equation (79) and the Lax pair equations (60) and (61), exact solutions of such an equation can be obtained and surface excitations can be studied [25].

For the quadratic potential case, the Gross–Pitaevskii equation, equation (65), describes a Bose gas trapped in a quadratic trap. This case has been addressed before by Liang *et al*, where the Lax pair was found for the special case of expulsive quadratic potential and scattering length growing exponentially with time. However, the Lax pair of Liang *et al* is a special case of our Lax Pair (for $c_2 = 0$ and $c_3 = 1$). In our case, the time dependence of the scattering length is not restricted to grow exponentially with time as in Liang's work. Instead, it can decay with time (for $c_2 = 1$ and $c_3 = 0$). It can also change sign around a divergent point. This can be obtained for $|c_2| < |c_3|$ and $c_2 c_3 < 0$. This behaviour for the scattering length resembles that of a Feschbach resonance. In the Feshbach resonance, the magnetic field is used to change the value and sign of the scattering length. The scattering length diverges at the point where the scattering length changes sign. Thus, if the magnetic field changes with time at a rate that equals λ_2, then equation (65) describes this situation and the exact solution of such an experimentally interesting case can be obtained using the Lax pair we found here together with the corresponding seed solution [25]. Furthermore, we can replace λ_2 by a pure imaginary parameter $\lambda_3 = \lambda_2 \mathrm{i}$. The potential in this case becomes impulsive and the scattering length $a(t)$ becomes sinusoidal. With proper choices for the arbitrary constants c_2 and c_3, the scattering length can be expressed in terms of $1/\sin(\lambda_3 t)$ or $1/\cos(\lambda_3 t)$ which are oscillatory but with periodic singularities.

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